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AFFINE CONNECTIONS AND DEFINING FUNCTIONS OF REAL HYPERSURFACES IN $\mathbf{C}^n$

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ABSTRACT. The affine connection and curvature introduced by Tanaka on a strongly pseudoconvex real hypersurface are computed explicitly in terms of its defining function. If Fefferman's defining function is used, then the Ricci form is shown to be a function multiple of the Levi form. The factor is computable by Fefferman's algorithm and its positivity implies the vanishing of certain cohomology groups (of the $\bar{\partial}_b$ complex) in the compact case.

Introduction. Let M be a strongly pseudoconvex real hypersurface in $\mathbf{C}^n$. Let θ be a nowhere vanishing real differential form on M which annihilates the complex tangent space at each point. It is well known that there exists a unique vector field ξ on M such that $\theta(\xi) = 1$ and $\xi \lrcorner d\theta = 0$. In [5], Tanaka has constructed a unique affine connection ∇ satisfying certain natural conditions with respect to ξ. Using this canonical affine connection associated to ξ (or θ), he obtained the following Bochner-Kodaira type formula for the boundary Laplacian $\Box_b$ (see [5, p. 47] and compare [4, p. 119]). Let $T'M$ be the bundle of (1, 0) vectors tangent to M. Then

$$\Box_b\phi = -\frac{n-q-1}{n-1}\sum_{\alpha=1}^{n-1}\nabla_{X_\alpha}\nabla_{\bar{X}_\alpha}\phi - \frac{q}{n-1}\sum_{\alpha=1}^{n-1}\nabla_{\bar{X}_\alpha}\nabla_{X_\alpha}\phi + \frac{n-q-1}{n-1}R_*(\phi), \qquad (*)$$

where ϕ is a smooth section of $\Lambda^q(\overline{T'M})^*$, $\{X_\alpha\}_{\alpha=1,\ldots,n-1}$ is a local orthonormal basis of $T'M$ with respect to the Levi metric defined by θ and R_* is the Ricci operator associated to ∇. In case M is compact, $(*)$ implies, by standard arguments, that if the selfadjoint operator R_* is positive definite everywhere, then there is no nonzero harmonic form ϕ on M and the cohomology groups $H^{0,q}(M)$ of the $\bar{\partial}_b$ complex vanish, for $q \neq 0, n-1$. $H^{0,q}(M)$ here coincides with $H^{0,q}(\mathfrak{B})$ in Kohn-Rossi [3, p. 466]. In passing, the above theory works for an abstractly given strongly pseudoconvex CR manifold, not necessarily imbedded as a real hypersurface in $\mathbf{C}^n$.

There is a choice of ξ (or θ) in the construction of ∇. One naturally asks if there exists a most natural choice. For any defining function f on M, one may take $\theta = -i\partial f$. Now Fefferman [1] has constructed special defining functions which

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satisfy a Monge-Ampere equation to order $< n + 1$ at M. The Monge-Ampere equation arises from considerations of the Bergman kernel, and is intimately related to the geometry of M. In this paper, we will show that if f satisfies the equation to second order at M, then for the affine connection associated to $\theta = -i\partial f$, the Ricci form is a function multiple of the Levi form. Thus, we obtain a real-valued function λ on M, computable using Fefferman's algorithm, such that the Ricci operator is positive definite at a point x in M if and only if $\lambda(x) > 0$. If λ is everywhere positive, then the cohomology groups $H^{0,q}(M)$ $(q \neq 0, n-1)$ vanish. Relevant to an open problem posed in **[1]** concerning intrinsic development, it would be interesting to know if there is some natural choice of θ for constructing ∇ on a CR manifold.

In §1, we state explicitly all the results from **[1]** and **[5]** which we need. In §2, we consider any defining function f and derive formulas for the affine connection and curvature associated to $\theta = -i\partial f$. It is interesting to compare our formulas to the local formulas of Riemannian and Kähler geometry as in **[2]**. We apply the formulas to consider the ellipsoids and to prove the main theorem in the last section. The formulas apply also to real hypersurfaces in complex manifolds; we shall return to their applications to isolated singularities (cf. **[5]**) later.

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1. Notations and preliminaries. Throughout this paper, we use the notation of tensor calculus. Small Greek (resp. Latin) indices always run from 1 to $n - 1$ (resp. 1 to n). Summation over repeated indices is understood. $z^1, \ldots, z^n$ are the coordinates of $\mathbf{C}^n$, and $f_j, f_{\bar{k}}, f_{j\bar{k}}, \ldots$ stand for the partial derivatives $\partial f/\partial z^j, \partial f/\partial \bar{z}^k, \partial^2 f/\partial z^j \partial \bar{z}^k, \ldots$. For a C^∞ manifold M, TM, T^*M, etc. have the usual meaning. For a vector bundle V over M, $\mathbf{C}V = V \otimes \mathbf{C}$, $\Gamma(V)$ denotes the space of C^∞ sections in V, and V_x denotes the fiber over $x \in M$.

Let M be a C^∞ strongly pseudoconvex real hypersurface in $\mathbf{C}^n$. Let $T'M$ (resp. $T''M$) be the subbundle of $\mathbf{C}TM$ consisting of vectors of type (1, 0) (resp. (0, 1)) in $\mathbf{C}^n$. There is a subbundle $H(M)$ of TM and a homomorphism I: $H(M) \to H(M)$ such that

(1) $\mathbf{C}H(M) = T'M \oplus T''M$.

(2) $I^2 = -1$ and $T'M = \{X - iIX | X \in H(M)\}$.

At each point $x \in M$, $H_x(M)$ is called the complex tangent space at x. Let $\theta \in \Gamma(T^*M)$ be any nowhere vanishing differential form on M which annihilates $H(M)$. There is a unique vector field ξ on M such that

$$\theta(\xi) = 1 \quad \text{and} \quad \xi \lrcorner d\theta = 0. \tag{3}$$

Extend I to a tensor field of type (1, 1) on M by setting $I(\xi) = 0$. Then,

$$I^2(X) = -X + \theta(X)\xi \quad \text{for any } X \in TM. \tag{4}$$

Let $\omega = -d\theta$. We shall consider the Levi form $\langle\, , \rangle$ defined by

$$\langle X, Y \rangle = \omega(IX, Y), \qquad X, Y \in \mathbf{C}T_xM, \quad x \in M.$$

Further we assume that $\langle\ ,\ \rangle$ is positive definite on $H(M)$. Let E be the 1-dimensional subbundle of $\mathbf{C}TM$ spanned by ξ. Thus, for each $x \in M$, the fiber $E_x = \mathbf{C}\xi_x$. For any $X \in \mathbf{C}TM$, we denote by $X_{T'}$ (resp. $X_{T''}$, X_E) the $T'M$- (resp. $T''M$-, E-) component of X with respect to the decomposition $\mathbf{C}TM = T'M \oplus T''M \oplus E$.

Now we recall Tanaka's canonical affine connection on M associated to ξ. This is a connection

$$\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M)$$

on M satisfying the following conditions:

(5) $\nabla_X \Gamma(H(M)) \subset \Gamma(H(M))$ for any $X \in \Gamma(TM)$ i.e. $H(M)$ is parallel.

(6) The tensor fields ξ, I, ω are parallel.

(7) Let T be the torsion of ∇. For any $X, Y \in \Gamma(H(M))$,

$$T(X, Y) = -\omega(X, Y)\xi \quad \text{and} \quad T(\xi, IY) = -IT(\xi, Y).$$

We shall use the following formulas [5, p. 31]. Let $X, Y \in \Gamma(T'M)$.

(8) $\nabla_{\overline{X}} Y = [\overline{X}, Y]_{T'}$.

(9) $\nabla_X Y \in \Gamma(T'M)$ and $\langle \nabla_X Y, \overline{W} \rangle = X\langle Y, \overline{W}\rangle - \langle Y, \overline{\nabla_{\overline{X}} W}\rangle$ for all $W \in \Gamma(T'M)$.

(10) $\nabla_\xi Y = L_\xi Y - \frac{1}{2} I(L_\xi I) Y$, where L_ξ denotes the Lie derivation.

Finally, we recall Fefferman's defining functions. Let

$$J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar 1} & \cdots & u_{\bar n} \\ u_1 & u_{1\bar 1} & \cdots & u_{1\bar n} \\ \vdots & \vdots & & \vdots \\ u_n & u_{n\bar 1} & \cdots & u_{n\bar n} \end{pmatrix}, \tag{11}$$

where u is any smooth function. In [1], Fefferman considered defining equations of M satisfying

(11)$'$ $J(u) = 1$, to high order at M.

He also found a clever algorithm for u up to order $n + 1$.

2. Formulas for connection and curvature. Let f be any defining function of M. Precisely, let f be a C^∞ real-valued function defined on some neighborhood of M such that M is defined by the equation $f = 0$, and $df \neq 0$ on M. Take $\theta = -i\partial f$. The assumption on $\langle\ ,\ \rangle$ implies that $f_{j\bar k} w^j \overline{w^k} > 0$ for any nonzero vector $w^j \partial/\partial z^j$ satisfying $w^j f_j = 0$. The matrix $(f_{j\bar k})$ need not be invertible. Let $\xi = \xi^j \partial/\partial z^j + \overline{\xi^j}\, \partial/\partial \bar z^j$. Condition (3) is equivalent to

$$i f_{\bar k} \overline{\xi^k} = 1 \tag*{(3)$_1$}$$

and

$$x^j f_j = 0 \quad \text{implies } x^j f_{j\bar k} \overline{\xi^k} = 0. \tag*{(3)$_2$}$$

Choose a local C^∞ orthonormal basis $\{X_\alpha = x_\alpha^j \partial/\partial z^j\}_{\alpha = 1, \ldots, n-1}$ of $T'M$ with respect to $\langle\ ,\ \rangle$. Thus,

$$x_\alpha^j f_j = 0 \tag{12}$$

and

$$x^j_\alpha f_{j\bar{k}} \overline{x^k_\beta} = \delta^\beta_\alpha. \tag{13}$$

PROPOSITION 1. *Let*

$$F = \begin{bmatrix} f & f_{\bar{1}} & \cdots & f_{\bar{n}} \\ f_1 & f_{1\bar{1}} & \cdots & f_{1\bar{n}} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n\bar{1}} & \cdots & f_{n\bar{n}} \end{bmatrix}$$

and

$$A = \begin{bmatrix} -\langle \xi, \xi \rangle' & -i\xi^1 & \cdots & -i\xi^n \\ i\overline{\xi^1} & a^{\bar{1}1} & \cdots & a^{\bar{1}n} \\ \vdots & \vdots & & \vdots \\ i\overline{\xi^n} & a^{\bar{n}1} & \cdots & a^{\bar{n}n} \end{bmatrix},$$

where $a^{\bar{j}k} = \overline{x^j_\alpha}\, x^k_\alpha$ *and* $\langle \xi, \xi \rangle' = \frac{1}{2}\langle \xi, \xi \rangle$. *Then* $FA = I_{n+1}$.

PROOF. $f_j dz^j$ and $f_{j\bar{k}} \overline{\xi^k} dz^j$ annihilate all X_α, hence they are linearly dependent. Since $df \neq 0$, we may assume that $f_{j\bar{k}} \overline{\xi^k} = a f_j$. Contracting with ξ^j gives $a = -i\langle \xi, \xi \rangle'$. Thus,

$$-f_j \langle \xi, \xi \rangle' + i f_{j\bar{k}} \overline{\xi^k} = 0. \tag{14}$$

Writing $\xi^j = x^j_n$, we have by $(3)_2$ and (13)

$$\begin{bmatrix} x^1_1 & \cdots & x^n_1 \\ \vdots & & \vdots \\ x^1_n & \cdots & x^n_n \end{bmatrix} \begin{bmatrix} f_{1\bar{1}} & \cdots & f_{1\bar{n}} \\ \vdots & & \vdots \\ f_{n\bar{1}} & \cdots & f_{n\bar{n}} \end{bmatrix} \begin{bmatrix} \overline{x^1_1} & \cdots & \overline{x^1_n} \\ \vdots & & \vdots \\ \overline{x^n_1} & \cdots & \overline{x^n_n} \end{bmatrix} = \begin{bmatrix} 1 & & & \mathbf{0} \\ & \ddots & & \\ & & 1 & \\ \mathbf{0} & & & \langle \xi, \xi \rangle' \end{bmatrix}. \tag{15}$$

(x^k_j) is invertible because $\xi^j \partial/\partial z^j \notin T'M$. Let $(y^k_j) = (x^k_j)^{-1}$, noting that $y^n_j = -if_j$ by $(3)_1$ and (12). Then

$$(f_{j\bar{l}})(\overline{x^l_r})(x^k_r) = (y^l_j) \begin{bmatrix} 1 & & & \mathbf{0} \\ & \ddots & & \\ & & 1 & \\ \mathbf{0} & & & \langle \xi, \xi \rangle' \end{bmatrix} (x^k_r). \tag{15'}$$

Now

$$f_{j\bar{l}}\,\overline{x_r^l}\;x_r^k = f_{j\bar{l}}\left(\overline{x_\alpha^l}\;x_\alpha^k + \overline{\xi^l}\;\xi^k\right) = f_{j\bar{l}}a^{\bar{l}k} - i\langle\xi,\xi\rangle' f_j\xi^k \quad \text{by (14)},$$

while the (j, k)-element on the right of (15)′ is

$$y_j^\alpha x_\alpha^k + \langle\xi,\xi\rangle' y_j^n x_n^k = \delta_j^k + (\langle\xi,\xi\rangle' - 1)y_j^n x_n^k = \delta_j^k + if_j\xi^k - i\langle\xi,\xi\rangle' f_j\xi^k.$$

Therefore, we get

$$-if_j\xi^k + f_{j\bar{l}}a^{\bar{l}k} = \delta_j^k. \tag{16}$$

By (12),

$$f_{\bar{j}}a^{\bar{j}k} = 0. \tag{17}$$

Since $f = 0$ on M, $(3)_1$, (14), (16), (17) prove the proposition.

We shall use the following notations:

$$F = \begin{bmatrix} f_{0\bar{0}} & \cdots & f_{0\bar{n}} \\ \vdots & & \vdots \\ f_{n\bar{0}} & \cdots & f_{n\bar{n}} \end{bmatrix}, \tag{18}$$

$$A = \begin{bmatrix} a^{\bar{0}0} & \cdots & a^{\bar{0}n} \\ \vdots & & \vdots \\ a^{\bar{n}0} & \cdots & a^{\bar{n}n} \end{bmatrix}. \tag{18$'$}$$

Proposition 1 implies that F is invertible in a neighborhood of M. Let $A = F^{-1}$ and use (18)′ to extend the functions $a^{\bar{k}l}$, $a^{\bar{k}0} = i\overline{\xi^k}$ and $a^{\bar{0}0} = a$. Then in a neighborhood of M, we have

$$af + if_{\bar{k}}\,\overline{\xi^k} = 1, \tag{3$)'_1$}$$

$$af_j + if_{j\bar{k}}\,\overline{\xi^k} = 0, \tag{14$'$}$$

$$-if_j\xi^l + f_{j\bar{k}}a^{\bar{k}l} = \delta_j^l, \tag{16$'$}$$

$$-if\xi^l + f_{\bar{k}}a^{\bar{k}l} = 0. \tag{17$'$}$$

Note that $\xi^j\partial/\partial z^j + \overline{\xi^j}\,\partial/\partial\overline{z^j}$ is not the vector field corresponding to $\theta = -i\partial f$ on each level real hypersurface unless $a = 0$.

We now give local formulas corresponding to (8), (9) and (10).

Proposition 2. *Let $X = x^j\partial/\partial z^j$ and $Y = y^j\partial/\partial z^j$ (with $x^jf_j = y^jf_j = 0$) be local C^∞ sections in $T'M$. Then*

(a) $\nabla_{\bar{X}}Y = (\bar{X}(y^k) + \overline{x^j}y^l\Gamma_{\bar{j}l}^k)\partial/\partial z^k$, *where* $\Gamma_{\bar{j}l}^k = -if_{\bar{j}l}\xi^k$.

(b) $\nabla_X Y = (X(y^k) + x^jy^l\Gamma_{jl}^k)\partial/\partial z^k$,

where $\Gamma_{jl}^k = \sum_{A=0}^n a^{\bar{A}k}(\partial/\partial z^j)f_{l\bar{A}} = \Gamma_{lj}^k$.

(c) $\nabla_\xi Y = \{\xi(y^k) - Y(\xi^k)\}\partial/\partial z^k$.

Proof. We first derive

$$\nabla_\xi Y = [\xi, Y]_{T'}. \tag{10$'$}$$

Recall $\theta(Y) = 0$ and $IY = iY$. Using (3), note that $\theta([\xi, Y]) = 0$. Then

$$\begin{aligned}\nabla_\xi Y &= [\xi, Y] + \tfrac{1}{2}I(I[\xi, Y] - [\xi, IY]) \quad \text{by (10)}\\ &= [\xi, Y] - \tfrac{1}{2}[\xi, Y] - \tfrac{i}{2}I[\xi, Y] \quad \text{by (4)}\\ &= \tfrac{1}{2}([\xi, Y]_{T'} + [\xi, Y]_{T''}) - \tfrac{i}{2}(i[\xi, Y]_{T'} - i[\xi, Y]_{T''}) = [\xi, Y]_{T'}.\end{aligned}$$

Next note that for any $Z = a^j\partial/\partial z^j + b^{\bar{j}}\partial/\partial \overline{z^j}$ in $\mathbf{C}TM$,

$$Z_E = \theta(Z)\xi, \qquad \theta(Z) = -ia^l f_l = ib^{\bar{l}} f_{\bar{l}}. \tag{19$_a$}$$

$$Z_{T'} = \left(a^k + ia^l f_l \xi^k\right)\frac{\partial}{\partial z^k}. \tag{19$_b$}$$

$$Z_{T''} = \left(b^{\bar{k}} - ib^{\bar{l}} f_{\bar{l}}\, \overline{\xi^k}\right)\frac{\partial}{\partial \overline{z^k}}. \tag{19$_c$}$$

(a) and (c) then follow easily from (8) and (10)$'$, using $\theta([\bar{X}, Y]) = i\langle Y, \bar{X}\rangle$ and $\theta([\xi, Y]) = 0$.

To get (b), observe that for any $W = w^i\partial/\partial z^j$ with $w^j f_j = 0$,

$$\begin{aligned}X\langle Y, \overline{W}\rangle - \langle Y, \overline{\nabla_{\bar{X}} W}\rangle &= X(y^k) f_{k\bar{m}}\overline{w^m} + x^j y^l f_{jl\bar{m}}\overline{w^m} \quad \text{since } y^j f_{j\bar{k}}\overline{\Gamma^k_{lm}} = 0\\ &= \left(X(y^k) + x^j y^l f_{jl\bar{i}} a^{\bar{i}k}\right) f_{k\bar{m}}\overline{w^m} \quad \text{by (16)}\\ &= \left\{X(y^k) + x^j y^l\left(a^{\bar{i}k} f_{jl\bar{i}} - i f_{jl}\xi^k\right)\right\} f_{k\bar{m}}\overline{w^m} \quad \text{by (3)}_2.\end{aligned}$$

Let $v^k = X(y^k) + x^j y^l(a^{\bar{i}k} f_{jl\bar{i}} - i f_{jl}\xi^k)$. By $(3)_1$ and (17), $v^k f_k = 0$, hence $v^k\partial/\partial z^k \in \Gamma(T'M)$. By (9), $\nabla_X Y = v^k\partial/\partial z^k$. The expression for Γ^k_{jl} simply follows from notations in (18) and (18)$'$, finishing the proof.

In the following, we consider $\Gamma^l_{\bar{j}k}$ and Γ^l_{jk} as functions on a neighborhood of M, defined by the formulas in Proposition 2.

Proposition 3. *Let $X = x^j\partial/\partial z^j$, $Y = y^j\partial/\partial z^j$ and $W = w^j\partial/\partial z^j$ be C^∞ sections in $T'M$. Then*

$$R(X, \bar{Y})W = (\nabla_X\nabla_{\bar{Y}} - \nabla_{\bar{Y}}\nabla_X - \nabla_{[X,\bar{Y}]})W = x^j\overline{y^k}w^l R^p_{j\bar{k}l}\frac{\partial}{\partial z^p},$$

where

$$R^p_{j\bar{k}l} = \frac{\partial}{\partial z^j}\Gamma^p_{\bar{k}l} + \frac{\partial}{\partial z^l}\Gamma^p_{\bar{k}j} - \frac{\partial}{\partial \overline{z^k}}\Gamma^p_{jl} + \Gamma^r_{\bar{k}l}\Gamma^p_{jr} + \Gamma^r_{\bar{k}j}\Gamma^p_{rl} - \Gamma^r_{jl}\Gamma^p_{\bar{k}r} + i f_{j\bar{k}l}\xi^p$$

and

$$R^p_{j\bar{k}l} f_p \equiv 0 \mod f_j, f_{\bar{k}}, f_l.$$

Proof. Decomposing $[X, \bar{Y}]$ by $(19)_{a,b,c}$ and using Proposition 2, one obtains by straightforward computation

$$\begin{aligned}R(X, \bar{Y})W = \Bigg\{ & x^j y^{\bar{k}} w^l\left(\frac{\partial}{\partial z^j}\Gamma^p_{\bar{k}l} - \frac{\partial}{\partial \overline{z^k}}\Gamma^p_{jl} + \Gamma^r_{\bar{k}l}\Gamma^p_{jr} - \Gamma^r_{jl}\Gamma^p_{\bar{k}r}\right)\\ & -i\langle X, \bar{Y}\rangle(\xi^r w^l\Gamma^p_{rl} + \overline{\xi^r}\, w^l\Gamma^p_{\bar{r}l} + W(\xi^p))\Bigg\}\frac{\partial}{\partial z^p}.\end{aligned}$$

Now

$$-i\langle X, \overline{Y}\rangle \xi^r w^l \Gamma^p_{rl} = x^j \overline{y^k}\, w^l \Gamma^r_{\bar{k}j}\Gamma^p_{rl}, \qquad \overline{\xi^r}\,\Gamma_{\bar{r}l} w^l = 0,$$

and

$$-i\langle X, \overline{Y}\rangle W(\xi^p) = x^j \overline{y^k}\,(-if_{j\bar{k}})W(\xi^p) = x^j y^{\bar{k}} w^l\left(\frac{\partial}{\partial z^l}\Gamma^p_{\bar{k}j} + if_{j\bar{k}l}\xi^l\right).$$

Hence

$$R(X, \overline{Y})W = x^j \overline{y^k}\, w^l R^p_{j\bar{k}l}\frac{\partial}{\partial z^p}$$

with $R^p_{j\bar{k}l}$ as given.

Next note that $R(X, \overline{Y})W \in \Gamma(T'M)$ for any $X, Y, W \in \Gamma(T'M)$. Hence $x^j\overline{y^k}w^l R^p_{j\bar{k}l}f_p = 0$ whenever $x^j f_j = y^j f_j = w^j f_j = 0$. The following lemma then implies that $R^p_{j\bar{k}l}f_p \equiv 0 \bmod f_j, f_{\bar{k}}, f_l$, finishing the proof of the proposition.

Lemma (Quotient law). *If $a_{j_1\cdots j_r\bar{k}_1\cdots\bar{k}_s}$ are n^{r+s} numbers such that $a_{j_1\cdots j_r\bar{k}_1\cdots\bar{k}_s}x_1^{j_1}\cdots x_r^{j_r}\overline{y_1^{k_1}\cdots y_s^{k_s}} = 0$ whenever $x_i^j f_j = 0$ $(i = 1, \ldots, r)$ and $y_l^k f_k = 0$ $(l = 1, \ldots, s)$, then $a_{j_1\cdots j_r\bar{k}_1\cdots\bar{k}_s} \equiv 0 \bmod f_{j_1}, \ldots, f_{j_r}, f_{\bar{k}_1}, \ldots, f_{\bar{k}_s}$.*

Proof. For simplicity we prove the case $r = s = 1$; the general case is similar. Since $df \neq 0$, we assume that $f_n \neq 0$. Consider

$$x = \Big(0, \ldots, \underset{(\alpha)}{1}, \ldots, 0, -f_\alpha/f_n\Big) \quad\text{and}\quad y = \Big(0, \ldots, \underset{(\beta)}{1}, \ldots, 0, -f_\beta/f_n\Big)$$

in $a_{j\bar{k}}x^j\overline{y^k}$. We get

$$a_{\alpha\bar{\beta}} = \frac{f_\alpha}{f_n}\,a_{n\bar{\beta}} + \frac{f_{\bar{\beta}}}{f_n}\,a_{\alpha\bar{n}} - \frac{f_\alpha f_{\bar{\beta}}}{f_n^2}\,a_{n\bar{n}}$$

and verify that

$$a_{j\bar{k}} = \frac{f_j}{f_n}\,a_{n\bar{k}} + \frac{f_{\bar{k}}}{f_n}\,a_{j\bar{n}} - \frac{f_j f_{\bar{k}}}{f_n^2}\,a_{n\bar{n}},$$

finishing the proof.

Proposition 4. *Let $X = x^j\partial/\partial z^j$, $Y = y^j\partial/\partial z^j$, $W = w^j\partial/\partial z^j$ and $U = u^j\partial/\partial z^j$ be C^∞ sections in $T'M$. Then*

$$\langle R(X, \overline{Y})W, \overline{U}\rangle = R_{j\bar{k}l\bar{m}}x^j\overline{y^k}\,w^l\overline{u^m},$$

where

$$R_{j\bar{k}l\bar{m}} = -f_{j\bar{k}l\bar{m}} + f_{jl\bar{r}}a^{\bar{r}s}f_{s\bar{k}\bar{m}} - i\big(f_{jl}f_{\bar{k}\bar{m}r}\xi^r - f_{\bar{k}\bar{m}}f_{jl\bar{s}}\overline{\xi^s}\big)$$
$$+\langle\xi, \xi\rangle'\big(f_{j\bar{k}}f_{l\bar{m}} + f_{l\bar{k}}f_{j\bar{m}} - f_{jl}f_{\bar{k}\bar{m}}\big).$$

PROOF. It suffices to compute $R^p_{\bar{k}l\bar{j}}f_{p\bar{m}}$ mod $f_{\bar{m}}$. There are seven terms:

$$\left(\frac{\partial}{\partial z^j}\Gamma^p_{\bar{k}l}\right)f_{p\bar{m}} = -\left(f_{j\bar{k}l}\xi^p f_{p\bar{m}} + f_{\bar{k}l}\frac{\partial \xi^p}{\partial z^j}f_{p\bar{m}}\right)$$

$$\equiv \langle \xi, \xi\rangle' f_{j\bar{m}}f_{l\bar{k}} + if_{\bar{k}l}f_{\bar{m}jp}\xi^p \text{ by (14) and (14)}'. \qquad \text{(i)}$$

$$\left(\frac{\partial}{\partial z^l}\Gamma^p_{\bar{k}j}\right)f_{p\bar{m}} = \langle \xi, \xi\rangle' f_{l\bar{m}}f_{j\bar{k}} + if_{\bar{k}j}f_{\bar{m}lp}\xi^p. \qquad \text{(i)}'$$

$$-\left(\frac{\partial}{\partial \bar{z}^k}\Gamma^p_{jl}\right)f_{p\bar{m}} = -f_{jl\bar{i}}\frac{\partial a^{\bar{i}p}}{\partial \bar{z}^k}f_{p\bar{m}} - f_{j\bar{k}l\bar{i}}a^{\bar{i}p}f_{p\bar{m}} + if_{jl}\frac{\partial \xi^p}{\partial \bar{z}^k}f_{p\bar{m}} + if_{jl\bar{k}}\xi^p f_{p\bar{m}}$$

$$\equiv f_{jl\bar{i}}a^{\bar{i}p}f_{p\bar{m}\bar{k}} + if_{jl\bar{i}}f_{\bar{k}\bar{m}}\xi^{\bar{i}} - f_{j\bar{k}l\bar{m}}$$

$$-\langle \xi, \xi\rangle' f_{jl}f_{\bar{m}\bar{k}} - if_{jl}f_{\bar{m}\bar{k}p}\xi^p \text{ by (14), (14)}', \text{(16), (16)}'. \qquad \text{(ii)}$$

By (14) and (16),

$$\Gamma^r_{\bar{k}l}\Gamma^p_{jr}f_{p\bar{m}} \equiv -if_{\bar{k}l}f_{\bar{m}jp}\xi^p, \qquad \text{(iii)}$$

$$\Gamma^r_{\bar{k}j}\Gamma^p_{rl}f_{p\bar{m}} \equiv -if_{\bar{k}j}f_{\bar{m}lp}\xi^p, \qquad \text{(iii)}'$$

$$-\Gamma^r_{jl}\Gamma^p_{\bar{k}r}f_{p\bar{m}} \equiv 0, \qquad \text{(iv)}$$

$$if_{j\bar{k}l}\xi^p f_{p\bar{m}} \equiv 0. \qquad \text{(v)}$$

The proposition follows immediately.

Observe that we have

$$R_{j\bar{k}l\bar{m}} = R_{j\bar{m}l\bar{k}} = R_{l\bar{k}j\bar{m}} = R_{l\bar{m}j\bar{k}}, \qquad (20)$$

$$\overline{R_{j\bar{k}l\bar{m}}} = R_{k\bar{j}m\bar{l}}, \qquad (21)$$

corresponding to the properties [5, p. 34]

$$\langle R(X, \bar{Y})W, \bar{U}\rangle = \langle R(X, \bar{U})W, \bar{Y}\rangle$$

$$= \langle R(W, \bar{Y})X, \bar{U}\rangle = \langle R(W, \bar{U})X, \bar{Y}\rangle, \qquad (20)'$$

$$\langle R(X, \bar{Y})W, \bar{U}\rangle = \langle W, \overline{R(Y, \bar{X})U}\rangle. \qquad (21)'$$

REMARKS. (a) Formulas for $R(\xi, \bar{Y})W$ etc. are complicated and will not be used. We shall however consider the following Ricci operator

$$R_*(Y) = \sum_{\alpha=1}^{n-1} R(X_\alpha, \bar{X}_\alpha)Y, \qquad Y \in \Gamma(T'M).$$

(b) For the real hyperquadric defined by $z^\alpha \overline{z^\alpha} + i(z^n - \overline{z^n})/2 = 0$, $\xi = 2(\partial/\partial z^n + \partial/\partial\overline{z^n})$ and $\langle \xi, \xi\rangle' = 0$, hence $R_{j\bar{k}l\bar{m}} = 0$. This is the flat case.

3. Applications of local formulas.

(A) *Ellipsoids*. Consider the ellipsoid E in $\mathbf{C}^n$ defined by the equation

$$f = a_{\alpha\beta}z^\alpha z^\beta + \overline{a_{\alpha\beta}}\ \overline{z^\alpha}\ \overline{z^\beta} + b_{\alpha\bar{\beta}}z^\alpha\overline{z^\beta}$$

$$+ az^n z^n + \bar{a}\,\overline{z^n}\ \overline{z^n} + bz^n\overline{z^n} - 1 = 0, \qquad (22)$$

where $a_{\alpha\beta}$ is symmetric, $b_{\alpha\bar{\beta}}$ $(=\overline{b_{\beta\bar{\alpha}}})$ is positive definite and b is positive. Clearly

$$R_{j\bar{k}l\bar{m}} = \langle \xi, \xi \rangle' (f_{j\bar{k}} f_{l\bar{m}} + f_{l\bar{k}} f_{j\bar{m}} - f_{jl} f_{\bar{k}\bar{m}}).$$

It is easy to solve for $\langle \xi, \xi \rangle'$ from the equation

$$\begin{bmatrix} 0 & f_{\bar{\beta}} & f_{\bar{n}} \\ f_\alpha & b_{\alpha\bar{\beta}} & 0 \\ f_n & 0 & b \end{bmatrix} \begin{bmatrix} -\langle \xi, \xi \rangle' \\ i\overline{\xi^\beta} \\ i\overline{\xi^n} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Setting $(c^{\bar{\alpha}\beta}) = (b_{\alpha\bar{\beta}})^{-1}$, one obtains

$$\langle \xi, \xi \rangle' \left(c^{\bar{\beta}\alpha} f_{\bar{\beta}} f_\alpha + b^{-1} |f_n|^2 \right) = 1.$$

Hence $\langle \xi, \xi \rangle' > 0$. A simple computation then shows that for any local section Y and any local orthonormal basis $\{X_\alpha\}$ of $T'M$,

$$\langle R_*(Y), \bar{Y} \rangle = \langle \xi, \xi \rangle' \sum_\alpha \left(\|X_\alpha\|_B^2 \|Y\|_B^2 + |\langle X_\alpha, \bar{Y} \rangle_B|^2 - |\langle X_\alpha, Y \rangle_A|^2 \right), \quad (23)$$

where $\langle Z, \bar{W} \rangle_B = b_{\beta\bar{\gamma}} z^\beta \overline{w^\gamma} + b z^n \overline{w^n}$, $\|Z\|_B^2 = \langle Z, \bar{Z} \rangle_B$ and $\langle Z, W \rangle_A = a_{\beta\gamma} z^\beta w^\gamma + a z^n w^n$.

Applying Tanaka's results, we obtain

PROPOSITION 5. *If $|a_{\beta\gamma}|$ and $|a|$ are small such that the right-hand side of* (23) *is everywhere positive for all Y, then there is no nonzero harmonic scalar form on M and the cohomology groups $H^{0,q}(E)$ $(q \neq 0, n-1)$ of the $\bar{\partial}_b$ complex vanish.*

The same method was applied to the sphere by Tanaka [5, p. 63]. Both cases are however very special examples of a result of Kohn and Rossi (obtained by a different method) which states that the same cohomology groups vanish for the boundary of any bounded strongly pseudoconvex domain in $\mathbf{C}^n$ [3, p. 467].

(B) We now consider special defining functions. Observe that for any defining function f of M, the equality

$$\frac{\partial}{\partial z^j} \log|\det F| = \sum_{B,C=0}^{n} a^{\bar{B}C} \frac{\partial}{\partial z^j} f_{C\bar{B}} \quad (24)$$

holds in a neighborhood of M.

THEOREM. (a) *If f satisfies $J(f) = \text{constant} + \mathcal{O}(f^s)$, then $\Gamma^l_{jl} = \mathcal{O}(f^{s-1})$.*
(b) *If f satisfies $J(f) = \text{constant} + \mathcal{O}(f^3)$, then*

$$\langle R_*(Y), \bar{Y} \rangle = -i \frac{\partial \xi^l}{\partial z^l} \langle Y, \bar{Y} \rangle \quad \textit{for any } Y \in \Gamma(T'M).$$

PROOF.

$$\Gamma^l_{jl} = a^{\bar{A}l} \frac{\partial}{\partial z^j} f_{l\bar{A}} = \frac{\partial}{\partial z^j} \log|\det F| - a^{\bar{A}0} \frac{\partial}{\partial z^j} f_{0\bar{A}}$$

by (24) and

$$-a^{\bar{A}0} \frac{\partial}{\partial z^j} f_{0\bar{A}} = a f_j + i f_{j\bar{k}} \overline{\xi^k} = 0$$

by (14)′. Hence

$$\Gamma^{l}_{jl} = \frac{\partial}{\partial z^{j}} \log|\det F| = \frac{\partial}{\partial z^{j}} \log|J(f)|,$$

and (a) follows. One easily verifies as in the proof of Proposition 4 that

$$R^{p}_{j\bar{k}l}f_{p} \equiv f_{j\bar{k}}f_{l}\langle \xi, \xi\rangle' \mod f_{j}, f_{\bar{k}}.$$

Setting $\tilde{R}^{p}_{j\bar{k}l} = R^{p}_{j\bar{k}l} + if_{j\bar{k}}f_{l}\langle \xi, \xi\rangle'\xi^{p}$, one has

$$R(X, \bar{Y})W = x^{j}\overline{y^{k}}\, w^{l}\tilde{R}^{p}_{j\bar{k}l}\frac{\partial}{\partial z^{p}},$$

where

$$\tilde{R}^{p}_{j\bar{k}l}f_{p} \equiv 0 \mod f_{j}, f_{\bar{k}}. \tag{25}$$

Now

$$\begin{aligned}\langle R_{*}(Y), \bar{Y}\rangle &= \sum_{\alpha}\langle R(Y, \bar{Y})X_{\alpha}, \overline{X_{\alpha}}\rangle \quad \text{by (20)}'\\ &= y^{j}\overline{y^{k}}\, x^{l}_{\alpha}\tilde{R}^{p}_{j\bar{k}l}f_{p\bar{m}}\overline{x^{m}_{\alpha}} = y^{j}\overline{y^{k}}\, \tilde{R}^{p}_{j\bar{k}l}f_{p\bar{m}}a^{\bar{m}l}\\ &= y^{j}\overline{y^{k}}\, \tilde{R}^{l}_{j\bar{k}l} \quad \text{by (16) and (25)}\\ &= y^{j}\overline{y^{k}}\left(\frac{\partial\Gamma^{l}_{\bar{k}l}}{\partial z^{j}} - if_{j\bar{k}}\frac{\partial\xi^{l}}{\partial z^{l}} - f_{j\bar{k}}\langle \xi, \xi\rangle'\right)\end{aligned}$$

using (a) and Proposition 3. One finishes the proof by noting that $\partial\Gamma^{l}_{\bar{k}l}/\partial z^{j} \equiv \langle \xi, \xi\rangle' f_{j\bar{k}} \mod f_{\bar{k}}$ by (14)′.

$-i\partial\xi^{l}/\partial z^{l}$ is the function λ referred to in the introduction.

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